Exercise 3.4

2. $y = \sqrt{4+3x}$

We can take $g(x)$ = 4x + 3 and $f(x) = \sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{4+3x} = \frac{d}{dx}\left((4+3x)^{\frac{1}{2}}\right) = \frac{1}{2}(4+3x)^{-\frac{1}{2}} \cdot \frac{d}{dx}(4x+3) = \frac{1}{2(4+3x)^{\frac{1}{2}}} \cdot 4 = \frac{2}{\sqrt{4+3x}}$$

7. $F(x) = (x^4 + 3x^2 - 2)^5$

$$\frac{dF}{dx} = \frac{d}{dx}((x^4+3x^2-2)^5) = 5(x^4+3x^2-2)^4 \cdot \frac{d}{dx}(x^4+3x^2-2) = 5(x^4+3x^2-2)^4(4x^3+6x)$$
$$= 10x(x^4+3x^2-2)^4(2x^2+3)$$

8. $F(x) = (4x - x^2)^{100}$

$$\frac{dF}{dx} = \frac{d}{dx}((4x-x^2)^{100}) = 100(4x-x^2)^{99} \cdot \frac{d}{dx}(4x-x^2) = 100(4x-x^2)^{99}(4-2x)$$
$$= 200(4x-x^2)^{99}(2-x)$$

9. $F(x) = \sqrt[4]{1+2x+x^3}$

$$\frac{dF}{dx} = \frac{d}{dx}(\sqrt[4]{1+2x+x^3}) = \frac{d}{dx}((1+2x+x^3)^{\frac{1}{4}}) = \frac{1}{4}(1+2x+x^3)^{-\frac{3}{4}} \cdot \frac{d}{dx}(1+2x+x^3)$$
$$= \frac{1}{4(1+2x+x^3)^{\frac{3}{4}}} \cdot (2+3x^2) = \frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

11. $g(t) = \dfrac{1}{(t^4+1)^3}$

$$\frac{dg}{dt} = \frac{d}{dt}\left(\frac{1}{(t^4+1)^3}\right) = \frac{d}{dt}\left((t^4+1)^{-3}\right) = -3(t^4+1)^{-4}\frac{d}{dt}\left(t^4+1\right) = -3(t^4+1)^{-4} \cdot 4t^3$$
$$= -\frac{12t^3}{(t^4+1)^4}$$

15. $y = xe^{-kx}$

$$\frac{dy}{dx} = \frac{d}{dx}(xe^{-kx}) = e^{-kx}\frac{d}{dx}(x) + x\frac{d}{dx}(e^{-kx}) = e^{-kx} \cdot 1 + x \cdot e^{-kx}\frac{d}{dx}(-kx) = e^{-kx} + xe^{-kx}(-k)$$
$$= e^{-kx} - kxe^{-kx} = e^{-kx}(1-kx)$$

18. $h(t) = (t^4 - 1)^3 (t^3 + 1)^4$

$$\frac{dh}{dt} = \frac{d}{dt}((t^4 - 1)^3 (t^3 + 1)^4) = (t^3 + 1)^4 \frac{d}{dt}((t^4 - 1)^3) + (t^4 - 1)^3 \frac{d}{dt}((t^3 + 1)^4)$$
$$= (t^3 + 1)^4 \cdot 3(t^4 - 1)^2 \frac{d}{dt}(t^4 - 1) + (t^4 - 1)^3 \cdot 4(t^3 + 1)^3 \frac{d}{dt}(t^3 + 1)$$
$$= 3(t^3 + 1)^4 (t^4 - 1)^2 \cdot 4t^3 + 4(t^4 - 1)^3 (t^3 + 1)^3 \cdot 3t^2$$
$$= 12t^3 (t^3 + 1)^4 (t^4 - 1)^2 + 12t^2 (t^4 - 1)^3 (t^3 + 1)^3$$
$$= 12t^2 (t^3 + 1)^3 (t^4 - 1)^2 (t(t^3 + 1) + t^4 - 1)$$
$$= 12t^2 (t^3 + 1)^3 (t^4 - 1)^2 (t^4 + t + t^4 - 1)$$
$$= 12t^2 (t^3 + 1)^3 (t^4 - 1)^2 (2t^4 + t - 1)$$

22. $y = e^{-5x} \cos 3x$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-5x} \cos 3x) = \cos 3x \frac{d}{dx}(e^{-5x}) + e^{-5x} \frac{d}{dx}(\cos 3x) = \cos 3x \cdot e^{-5x} \frac{d}{dx}(-5x) + e^{-5x} \cdot (-\sin 3x) \frac{d}{dx}(3x)$$
$$= \cos 3x e^{-5x}(-5) - \sin 3x e^{-5x} \cdot 3 = -5e^{-5x} \cos 3x - 3e^{-5x} \sin 3x$$
$$= -e^{-5x}(5 \cos 3x + 3 \sin 3x)$$

25. $F(z) = \sqrt{\frac{z-1}{z+1}}$

$$\frac{dF}{dz} = \frac{d}{dz}\left(\sqrt{\frac{z-1}{z+1}}\right) = \frac{d}{dz}\left(\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right) = \frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}} \frac{d}{dz}\left(\frac{z-1}{z+1}\right)$$

$$= \frac{1}{2}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} \frac{(z+1)\frac{d}{dz}(z-1) - (z-1)\frac{d}{dz}(z+1)}{(z+1)^2}$$
$$= \frac{1}{2}\sqrt{\frac{z+1}{z-1}} \frac{(z+1)\cdot 1 - (z-1)\cdot 1}{(z+1)^2} = \frac{1}{2}\sqrt{\frac{z+1}{z-1}} \frac{z+1-z+1}{(z+1)^2} = \frac{1}{2}\sqrt{\frac{z+1}{z-1}} \frac{2}{(z+1)^2}$$
$$= \sqrt{\frac{z+1}{z-1}} \frac{1}{(z+1)^2} = \frac{1}{\sqrt{z-1}(z+1)^{\frac{3}{2}}} = \frac{1}{(z+1)\sqrt{(z-1)(z+1)}}$$

32. $y = \tan^2(3\theta)$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\tan^2(3\theta)) = 2\tan(3\theta)\frac{d}{d\theta}(\tan(3\theta)) = 2\tan(3\theta) \cdot \sec^2(3\theta)\frac{d}{d\theta}(3\theta)$$
$$= 2\tan(3\theta)\sec^2(3\theta) \cdot 3 = 6\tan(3\theta)\sec^2(3\theta)$$

33. $y = \sec^2 x + \tan^2 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec^2 x + \tan^2 x) = \frac{d}{dx}(\sec^2 x) + \frac{d}{dx}(\tan^2 x) = 2\sec x \frac{d}{dx}(\sec x) + 2\tan x \frac{d}{dx}(\tan x)$$
$$= 2\sec x \cdot \sec x \tan x + 2\tan x \cdot \sec^2 x = 4\sec^2 x \tan x$$

52. $y = \sin x + \sin^2 x$, (0, 0)

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + \sin^2 x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\sin^2 x) = \cos x + 2\sin x \frac{d}{dx}(\sin x) = \cos x + 2\sin x \cos x$$
$$= \cos x + \sin 2x$$

The slope of the tangent line at the point (0, 0) is

$$m = \left.\frac{dy}{dx}\right|_{x=0} = \cos 0 + \sin(2 \cdot 0) = 1$$

Use the point-slope form of the equation of a line

$y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$

$y = x$

63.

(a) $h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

Then

$h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$

(b) $H(x) = g(f(x))$

$H'(x) = g'(f(x)) \cdot f'(x)$

Then

$H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$

Also, find the derivative of

(a) 4^x

$$\frac{d}{dx}(4^x) = 4^x \ln 4$$

(b) 7^x

$$\frac{d}{dx}(7^x) = 7^x \ln 7$$

(c) 3^{2x}

$$\frac{d}{dx}(3^{2x}) = 3^{2x} \ln 3 \cdot \frac{d}{dx}(2x) = 3^{2x} \ln 3 \cdot 2 = (2\ln 3)3^{2x}$$

(d) 5^{6x}

$$\frac{d}{dx}(5^{6x}) = 5^{6x}\ln 5 \cdot \frac{d}{dx}(6x) = 5^{6x}\ln 5 \cdot 6 = (6\ln 5)5^{6x}$$

Exercise 3.5

5. $x^3 + y^3 = 1$

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = -\frac{3x^2}{3y^2}$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

7. $x^2 + xy - y^2 = 4$

$$\frac{d}{dx}(x^2 + xy - y^2) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^2) = 0$$

$$2x + y\frac{d}{dx}(x) + x\frac{d}{dx}(y) - \frac{d}{dx}(y^2) = 0$$

$$2x + y \cdot 1 + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$2x + y + \frac{dy}{dx}(x - 2y) = 0$$

$$\frac{dy}{dx}(x - 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x - 2y}$$

$$\frac{dy}{dx} = -\frac{2x + y}{x - 2y}$$

11. $x^2y^2 + x\sin y = 4$

$$\frac{d}{dx}(x^2y^2 + x\sin y) = \frac{d}{dx}(4)$$
$$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(x\sin y) = 0$$
$$y^2\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(y^2) + \sin y\frac{d}{dx}(x) + x\frac{d}{dx}(\sin y) = 0$$
$$y^2 \cdot 2x + x^2 \cdot 2y\frac{dy}{dx} + \sin y \cdot 1 + x \cdot \cos y\frac{dy}{dx} = 0$$
$$2xy^2 + 2x^2y\frac{dy}{dx} + \sin y + x\cos y\frac{dy}{dx} = 0$$
$$2xy^2 + \sin y + \frac{dy}{dx}(2x^2y + x\cos y) = 0$$
$$\frac{dy}{dx}(2x^2y + x\cos y) = -2xy^2 - \sin y$$
$$\frac{dy}{dx} = \frac{-2xy^2 - \sin y}{2x^2y + x\cos y}$$
$$\frac{dy}{dx} = -\frac{2xy^2 + \sin y}{2x^2y + x\cos y}$$

13. $4\cos x\sin y = 1$

$$\frac{d}{dx}(4\cos x\sin y) = \frac{d}{dx}(1)$$
$$4\frac{d}{dx}(\cos x\sin y) = 0$$
$$\frac{d}{dx}(\cos x\sin y) = 0$$
$$\sin y\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\sin y) = 0$$
$$\sin y(-\sin x) + \cos x\cos y\frac{dy}{dx} = 0$$
$$\cos x\cos y\frac{dy}{dx} = \sin x\sin y$$
$$\frac{dy}{dx} = \frac{\sin x\sin y}{\cos x\cos y}$$
$$\frac{dy}{dx} = \tan x\tan y$$

25. $x^2 + xy + y^2 = 3$, $(1, 1)$

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3)$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$
$$2x + y\frac{d}{dx}(x) + x\frac{d}{dx}(y) + 2y\frac{dy}{dx} = 0$$
$$2x + y \cdot 1 + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$2x + y + \frac{dy}{dx}(x + 2y) = 0$$
$$\frac{dy}{dx}(x + 2y) = -2x - y$$
$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

The slope of the tangent line at the point (1, 1) is

$$m = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{-2(1) - 1}{1 + 2(1)} = -1$$

Use the point-slope form of the equation of a line

$y - y_1 = m(x - x_1)$
$y - 1 = -1(x - 1)$
$y = -x + 1 + 1$
$y = -x + 2$

31

(a). $y^2 = 5x^4 - x^2$, (1, 2)

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(5x^4 - x^2)$$
$$2y\frac{d}{dx}(y) = \frac{d}{dx}(5x^4) - \frac{d}{dx}(x^2)$$
$$2y\frac{dy}{dx} = 20x^3 - 2x$$
$$\frac{dy}{dx} = \frac{20x^3 - 2x}{2y}$$
$$\frac{dy}{dx} = \frac{10x^3 - x}{y}$$

The slope of the tangent line at the point (1, 2) is

$$m = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=2}} = \frac{10(1)^3 - 1}{2} = \frac{9}{2}$$

Use the point-slope form of the equation of a line

$y - y_1 = m(x - x_1)$

$$y - 2 = \frac{9}{2}(x - 1)$$

$$y = \frac{9}{2}x - \frac{9}{2} + 2$$

$$y = \frac{9}{2}x - \frac{5}{2}$$

33. $9x^2 + y^2 = 9$

Find the first derivative

$$\frac{d}{dx}(9x^2 + y^2) = \frac{d}{dx}(9)$$

$$\frac{d}{dx}(9x^2) + \frac{d}{dx}(y^2) = 0$$

$$18x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -18x$$

$$\frac{dy}{dx} = -\frac{9x}{y}$$

Find the second derivative

$$\frac{d^2y}{dx^2} = -\frac{y\frac{d}{dx}(9x) - 9x\frac{d}{dx}(y)}{y^2} = -\frac{y \cdot 9 - 9x\frac{dy}{dx}}{y^2} = -\frac{9y - 9x \cdot \left(-\frac{9x}{y}\right)}{y^2}$$

$$= -\frac{9y + \frac{81x^2}{y}}{y^2} = -\frac{9y^2 + 81x^2}{y^3} = -\frac{9(9x^2 + y^2)}{y^3} = -\frac{9 \cdot 9}{y^3} = -\frac{81}{y^3}$$

45. $y = \tan^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sqrt{x}) = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x}) = \frac{1}{1 + x} \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{1 + x} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2(1 + x)\sqrt{x}}$$

47. $y = \sin^{-1}(2x+1)$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(2x+1)) = \frac{1}{\sqrt{1-(2x+1)^2}}\frac{d}{dx}(2x+1) = \frac{1}{\sqrt{1-4x^2-4x-1}} \cdot 2 = \frac{2}{\sqrt{-4x^2-4x}}$$

$$= \frac{2}{\sqrt{4(-x^2-x)}} = \frac{2}{2\sqrt{-x^2-x}} = \frac{1}{\sqrt{-x^2-x}}$$

53. $y = \cos^{-1}(e^{2x})$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}(e^{2x})) = -\frac{1}{\sqrt{1-(e^{2x})^2}}\frac{d}{dx}(e^{2x}) = -\frac{1}{\sqrt{1-e^{4x}}} \cdot e^{2x}\frac{d}{dx}(2x)$$

$$= -\frac{e^{2x}}{\sqrt{1-e^{4x}}} \cdot 2 = -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

64. $x^2 - xy + y^2 = 3$, (–1, 1)

(a).

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$2x - y\frac{d}{dx}(x) - x\frac{d}{dx}(y) + 2y\frac{dy}{dx} = 0$$

$$2x - y \cdot 1 - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x - y + \frac{dy}{dx}(-x + 2y) = 0$$

$$\frac{dy}{dx}(2y - x) = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

The slope of the tangent line at the point (–1, 1) is

$$m_1 = \left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{1-2(-1)}{2(1)-(-1)} = 1$$

The slope of the normal line at the point (–1, 1) is

$$m_2 = -\frac{1}{m_1} = -\frac{1}{1} = -1$$

Use the point-slope form of the equation of a line

$y - y_1 = m(x - x_1)$

$y - 1 = -1(x - (-1))$

$y - 1 = -1(x + 1)$

$y = -x - 1 + 1$

$y = -x$

Find the point of intersection of the normal line $y = -x$ with the ellipse
$y = -x$
$x^2 - xy + y^2 = 3$

$x^2 - x(-x) + (-x)^2 = 3$
$3x^2 = 3$
$x = \pm 1$
Find y for $x = 1$:
$y = -1$
The second point of intersection is (1, –1).

Exercise 1.6

59.

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $-\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}$

(b) $\cos^{-1}(-1) = \pi$ because $\cos\pi = -1$ and $0 \le \pi \le \pi$

61.

(a) $\arctan 1 = \frac{\pi}{4}$ because $\tan\frac{\pi}{4} = 1$ and $-\frac{\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}$

(b) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ because $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$

66. $\tan(\sin^{-1} x)$

Since

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sin x}{\sqrt{1-\sin^2 x}}$$

then

$$\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\sqrt{1-\sin^2(\sin^{-1} x)}} = \frac{x}{\sqrt{1-(\sin(\sin^{-1} x))^2}} = \frac{x}{\sqrt{1-x^2}}$$