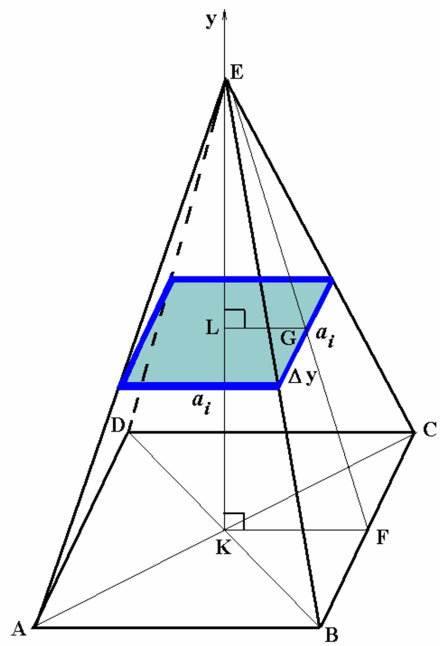
1. A pyramid, with a square base of side length 150 meters, is 100 meters tall. If the mass of the stone used in construction is 2000 kg. per cubic meter, find the total work done in raising the rock for the pyramid from ground level to its vertical height (consider only the vertical movement—ignore any horizontal movement



Let's measure height from the base of the pyramid by introducing a vertical coordinate line. The pyramid extends from 0 to 100 m and we divide the interval [0, 100] into *n* subintervals with endpoints $y_0, y_1, y_2, \ldots, y_n$ and choose y_i^* in the i^{th} subinterval. This divides the pyramid into *n* layers. The i^{th} layer is approximated by a square with side a_i and height Δy. From the similar triangles EKF and ELG we get

$$\frac{KF}{EK} = \frac{LG}{EL}$$

Since EL = EK – LK then

$$\frac{KF}{EK} = \frac{LG}{EK - LK}$$

$$\frac{75}{100} = \frac{\frac{1}{2}a_i}{100 - y_i^*}$$

$$\frac{3}{4} = \frac{\frac{1}{2}a_i}{100 - y_i^*}$$

$$\frac{1}{2}a_i = \frac{3}{4}(100 - y_i^*)$$

$$a_i = \frac{3}{2}(100 - y_i^*)$$

Thus an approximation to the volume of the i^{th} layer of rock is

$$V_i \approx a_i^2 \Delta y = \left(\frac{3}{2}(100 - y_i^*)\right)^2 \Delta y = \frac{9}{4}(100 - y_i^*)^2 \Delta y$$

its mass is

$$m_i = \rho V_i = \frac{9}{4}\rho(100 - y_i^*)^2 \Delta y$$

The force required to raise this layer must overcome the force of gravity

$$F_i = m_i g = \frac{9}{4}\rho g(100 - y_i^*)^2 \Delta y$$

Each particle in the layer must travel a distance of approximately y_i^*. The work done to raise this layer to the height y_i^* is

$$W_i = F_i y_i^* = \frac{9}{4}\rho g y_i^*(100 - y_i^*)^2 \Delta y$$

To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} W_i = \lim_{n\to\infty}\frac{9}{4}\rho g y_i^*(100-y_i^*)^2\Delta y = \int_0^{100}\frac{9}{4}\rho g y(100-y)^2 dy$$

$$= \frac{9}{4}\rho g\int_0^{100} y(10000-200y+y^2)dy = \frac{9}{4}\rho g\int_0^{100}(10000y-200y^2+y^3)dy$$

$$= \frac{9}{4}\rho g\left(5000y^2 - \frac{200}{3}y^3 + \frac{1}{4}y^4\right)\Bigg|_0^{100}$$

$$= \frac{9}{4}\rho g\left(5000\cdot 100^2 - \frac{200}{3}\cdot 100^3 + \frac{1}{4}\cdot 100^4\right)$$

$$= \frac{9}{4}\rho g\left(5\cdot 10^7 - \frac{2}{3}\cdot 10^8 + \frac{1}{4}\cdot 10^8\right)$$

$$= \frac{9}{4}\rho g\left(5 - \frac{20}{3} + \frac{10}{4}\right)\cdot 10^7 = \frac{9}{4}\cdot 2000(9.8)\left(\frac{60}{12} - \frac{80}{12} + \frac{30}{12}\right)\cdot 10^7$$

$$= \frac{9}{4}\cdot 2000(9.8)\left(\frac{60}{12} - \frac{80}{12} + \frac{30}{12}\right)\cdot 10^7$$

$$= 3.675\times 10^{11} \quad J$$

2. Find the volume of the pyramid in problem 1. Derive this using cross-sectional areas and integration.

From the solution of the problem #1 we have

$$V_i \approx a_i^2\Delta y = \left(\frac{3}{2}(100-y_i^*)\right)^2 \Delta y = \frac{9}{4}(100-y_i^*)^2\Delta y$$

To find the total volume, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} V_i = \lim_{n\to\infty}\frac{9}{4}(100-y_i^*)^2\Delta y = \int_0^{100}\frac{9}{4}(100-y)^2 dy$$

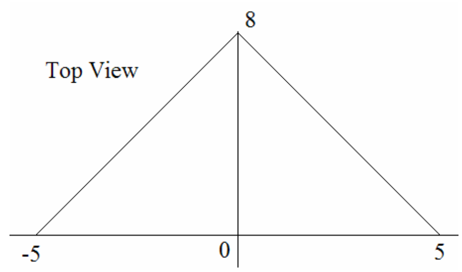
$$= \frac{9}{4}\int_0^{100}(10000-200y+y^2)dy = \frac{9}{4}\left[10000y - 100y^2 + \frac{1}{3}y^3\right]_0^{100}$$

$$= \frac{9}{4}\left(10000\cdot 100 - 100\cdot 100^2 + \frac{1}{3}\cdot 100^3\right)$$

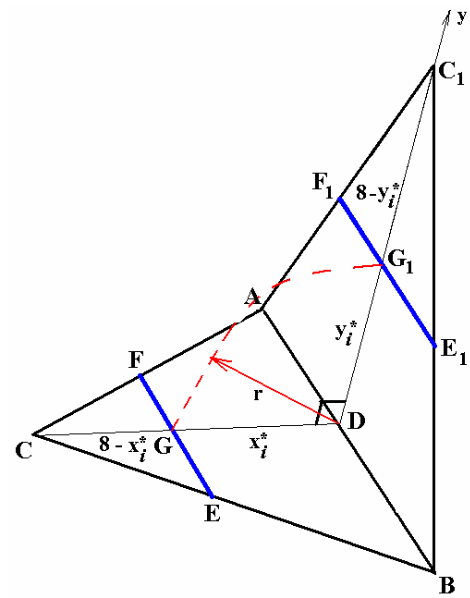
$$= \frac{9}{4}\left(10 - 10 + \frac{10}{3}\right)\times 10^5$$

$$= 7.5\times 10^5 \quad m^3$$

3. A triangular steel plate in the shape of an isosceles triangle is lying on the ground. The base (longest of the three sides) is 10 ft. long, and the height of the triangle is 8 ft. The triangle is raised to a vertical position leaving the 10 ft. base side on the ground. If the steel weighs one pound per square foot, find the work done in raising the plate to a vertical position.



When we raise the triangle each its point moves along a quarter-circle with radius r equal to the distance from the point to the axis of rotation AB. For example, the point G will be displaced to the point G_1. Consider the line segment FE parallel to the base AB of triangle ABC. It will be displaced to the line segment F_1E_1. Choose a coordinate system with the y-axis vertical. The triangle ABC_1 extends from 0 to 8 ft. We divide the interval [0, 8] into *n* subintervals with endpoints y_0, $y_1, y_2, \ldots, y_n$ and choose y_i^* in the i^{th} subinterval. This divides the triangle into *n* strips. The i^{th} strip is approximated by a rectangle with side a_i and height Δy. From the similar triangles ABC_1 and $F_1E_1C_1$ we have



$$\frac{C_1G_1}{C_1D} = \frac{F_1E_1}{AB} \quad \Rightarrow \quad \frac{8-y_i^*}{8} = \frac{a_i}{10}$$

$$a_i = \frac{5}{4}(8-y_i^*)$$

The strip F_1E_1 can be approximated as a rectangle, so its area is

$$A_i \approx a_i \Delta y = \frac{5}{4}(8-y_i^*)\Delta y$$

The weight of the strip is

$$w_i = \rho A_i \approx \frac{5}{4}\rho(8-y_i^*)\Delta y$$

where ρ is the density.

This strip is raised at the height y_i^*, so the work is

$$W_i = y_i^* w_i \approx \frac{5}{4}\rho y_i^*(8-y_i^*)\Delta y$$

To find the total work done in raising the triangle, we add the contributions of each of the *n* strips and then take the limit as $n \to \infty$:

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} W_i = \lim_{n\to\infty}\frac{5}{4}\rho y_i^*(8-y_i^*)\Delta y = \int_0^8 \frac{5}{4}\rho y(8-y)dy$$

$$= \frac{5}{4}\rho\int_0^8 y(8-y)dy = \frac{5}{4}\rho\int_0^8 (8y-y^2)dy$$

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} W_i = \lim_{n\to\infty}\frac{5}{4}\rho y_i^*(8-y_i^*)\Delta y = \int_0^8 \frac{5}{4}\rho y(8-y)dy$$

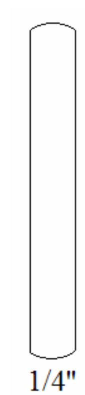
$$= \frac{5}{4}\rho\left[4y^2 - \frac{1}{3}y^3\right]_0^8 = \frac{5}{4}\rho\left(4\cdot 8^2 - \frac{1}{3}\cdot 8^3\right)$$

$$= \frac{5}{4}\rho\left(256 - \frac{512}{3}\right) = \frac{5}{4}\rho\cdot\frac{256}{3}$$

$$= \frac{320}{3}\rho = \frac{320}{3}\cdot 1$$

$$= \frac{320}{3}\ lb\ \ ft$$

4. A ring is generated by drilling a hole of radius r through the center of a sphere of radius R. If the width of the ring is ¼ inch, find the volume of the ring. Does the volume depend on R? on r?



We center the sphere at the origin so its equation is $x^2 + y^2 + z^2 = R^2$, and the equation of the cylinder is $x^2 + z^2 = r^2$. Consider the cross-section of the sphere by the plane z = 0. We observe that the ring may be obtained by rotation of the region bounded by a vertical line x = r and a semicircle $x = \sqrt{R^2 - y^2}$ about the y-axis. A cross-section perpendicular to the y-axis is a washer with inner radius r and outer radius $x = \sqrt{R^2 - y^2}$. The cross-sectional aria is

$$A(y) = \pi\left(\sqrt{R^2 - y^2}\right)^2 - \pi r^2 = \pi\left(R^2 - y^2\right) - \pi r^2 = \pi(R^2 - y^2 - r^2)$$

The ring lies between $y = -\frac{h}{2}$ and $y = \frac{h}{2}$ so its volume is

$$V = \int_{-h/2}^{h/2} A(y)dy = \int_{-h/2}^{h/2} \pi(R^2 - y^2 - r^2)dy = \pi\int_{-h/2}^{h/2}(R^2 - y^2 - r^2)dy$$

$$= \pi\left[R^2 y - \frac{1}{3}y^3 - r^2 y\right]_{-h/2}^{h/2}$$

$$= \pi\left(R^2\cdot\frac{h}{2} - \frac{1}{3}\left(\frac{h}{2}\right)^3 - r^2\left(\frac{h}{2}\right) - \left(R^2\left(-\frac{h}{2}\right) - \frac{1}{3}\left(-\frac{h}{2}\right)^3 - r^2\left(-\frac{h}{2}\right)\right)\right)$$

$$= \pi\left(\frac{1}{2}R^2 h - \frac{1}{24}h^3 - \frac{1}{2}r^2 h + \frac{1}{2}R^2 h - -\frac{1}{24}h^3 - \frac{1}{2}r^2 h\right)$$

$$= \pi\left(R^2h - \frac{1}{12}h^3 - r^2h\right)$$

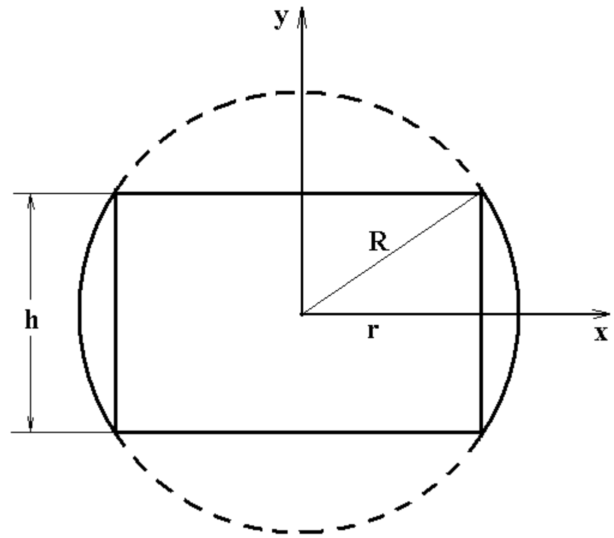
$$= \pi\left((R^2 - r^2)h - \frac{1}{12}h^3\right)$$

$$= \pi\left(\left(\frac{h}{2}\right)^2 h - \frac{1}{12}h^3\right)$$

$$= \pi\left(\frac{1}{4}h^3 - \frac{1}{12}h^3\right)$$

$$= \frac{1}{6}\pi h^3$$

We can see that the volume of the ring does not depend either on R or r.



5. Prove the following: If f and g are integrable on [a,b], and f(x) $\leq$ g(x) on [a,b], then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. Use the definition of the definite integral, and show and give an reason for each step.

We are given that $f(x) \leq g(x)$ for all x in $[a, b]$

In particular, $f(x_i) \leq g(x_i)$ for all i.

Multiplying by Δx, we get

$f(x_i)\Delta x \leq g(x_i)\Delta x$

Adding, we get

$$\sum_{i=1}^{n} f(x_i)\Delta x \leq \sum_{i=1}^{n} g(x_i)\Delta x$$

Taking the limits,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \leq \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x$$

Using the definition of definite integrals,

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

6. Compute

$$\int \frac{\sin(2x)dx}{1+\sin^4(x)} = \int \frac{2\sin(x)\cos(x)dx}{1+\sin^4(x)}$$

$$\left. \begin{array}{l} u = \sin(x) \\ du = \cos(x)dx \end{array} \right\|$$

$$= \int \frac{2udu}{1+u^4}$$

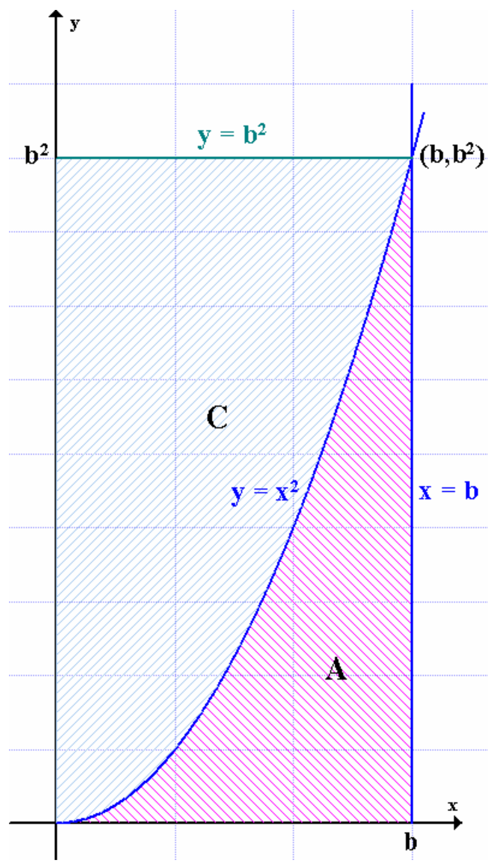
$$\left. \begin{array}{l} v = u^2 \\ dv = 2udu \end{array} \right\|$$

$$= \int \frac{dv}{1+v^2} = \tan^{-1} v + C = \tan^{-1} u^2 + C$$

$$= \tan^{-1}(\sin^2(x)) + C$$

7. Let A be the region bounded by y = x^2, y = 0, and x = b. for b > 0. Let C be the region in the first quadrant bounded by y = x^2, x = 0, and y = b^2.

a) Is there a value of b that makes the areas of the two regions equal?



The area of the region A is

$$A_1 = \int_0^b x^2 dx = \left[\frac{1}{3}x^3\right]_0^b = \frac{1}{3}b^3$$

The area of the region C is

$$A_1 = \int_0^b (b^2 - x^2)dx = \left[b^2 x - \frac{1}{3}x^3\right]_0^b$$

$$= b^3 - \frac{1}{3}b^3 = \frac{2}{3}b^3$$

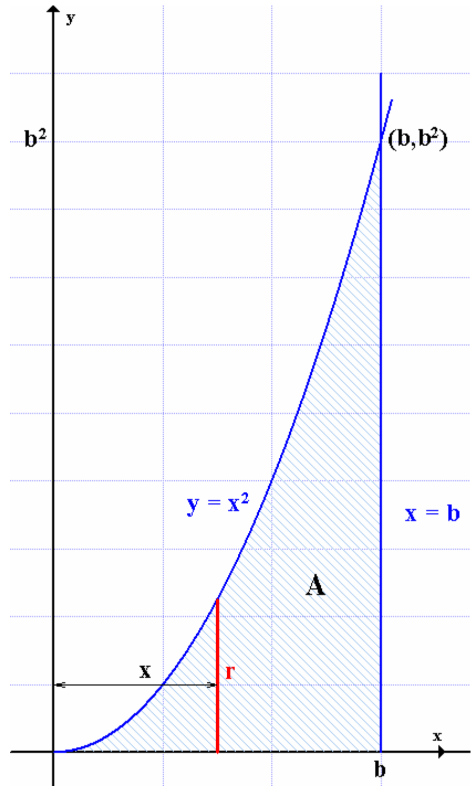
Find when both areas are the same

$$A_1 = A_2$$

$$\frac{1}{3}b^3 = \frac{2}{3}b^3$$

This equation has the only solution b = 0, but in this case there are no regions nor areas, so the solution has no physical sense.

b) Is there a value of b that makes region A sweep out the same volume when revolved about the x-axis and the y-axis?



Find the volume of the solid formed by rotating the region A about the x-axis:

When we slice through the point x, we get a disk with radius $r = x^2$. The area of this cross-section is

$A(x) = \pi r^2 = \pi(x^2)^2 = \pi x^4$

And the volume of the approximating cylinder is

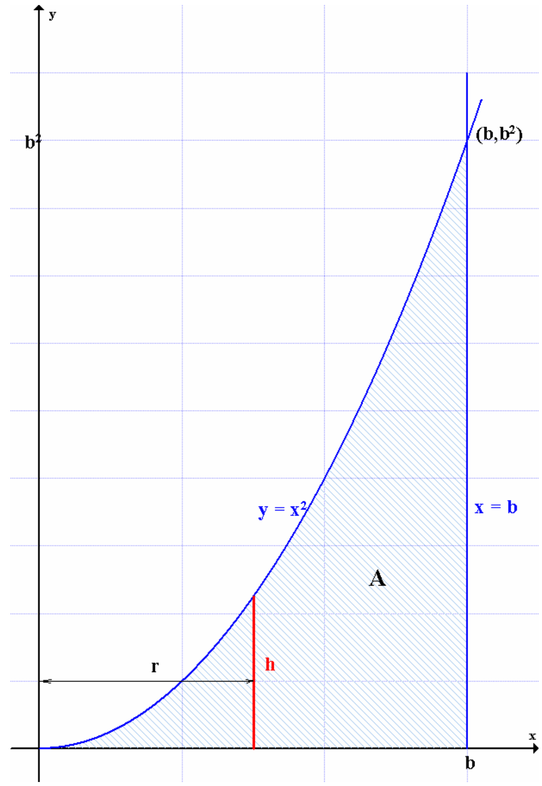
$A(x)\Delta x = \pi x^4 \Delta x$

The solid lies between x = 0 and x = b, so its volume is

$$V_1 = \int_0^b A(x)dx = \int_0^b \pi x^4 dx = \left[\frac{1}{5}\pi x^5\right]_0^b = \frac{1}{5}\pi b^5$$

Find the volume of the solid formed by rotating the region A about the y-axis:

A typical shell has radius r = x, circumference $2\pi r = 2\pi x$ and height $h = x^2$. By the shell method, the volume is



$$V_2 = \int_0^b 2\pi rh\,dx = \int_0^b 2\pi x x^2\,dx$$

$$= 2\pi\int_0^b x^3 dx = 2\pi\left[\frac{1}{4}x^4\right]_0^b$$

$$= 2\pi\cdot\frac{1}{4}b^4 = \frac{1}{2}\pi b^4$$

Find b when both volumes are equal:

$V_1 = V_2$

$$\frac{1}{5}\pi b^5 = \frac{1}{2}\pi b^4$$

$$\frac{1}{5}\pi b^5 - \frac{1}{2}\pi b^4 = 0$$

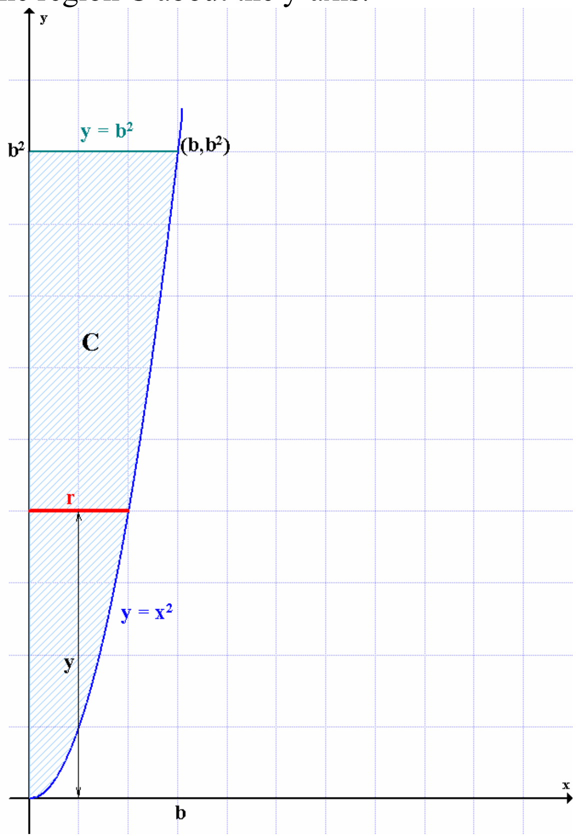
$$\frac{1}{10}\pi b^4(2b-5) = 0$$

$$b = 0 \quad or \quad b = \frac{5}{2}$$

Again, the solution b = 0 has no physical sense, so the only solution is $b = \frac{5}{2}$.

c) For what values of b will regions A and C sweep out the same volume, when revolved about the y-axis?

Find the volume of the solid formed by rotating the region C about the y-axis:



When we slice through the point y, we get a disk with radius $r = x = \sqrt{y}$. The area of this cross-section is

$A(y) = \pi r^2 = \pi(\sqrt{y})^2 = \pi y$

And the volume of the approximating cylinder is

$A(y)\Delta y = \pi y \Delta y$

The solid lies between y = 0 and y = b^2, so its volume is

$$V_3 = \int_0^{b^2} A(y)dy = \int_0^{b} \pi y dy = \left[\frac{1}{2}\pi y^2\right]_0^{b^2} = \frac{1}{2}\pi b^4$$

From part (b) the volume of the solid formed by rotating the region A about the y-axis is

$$V_2 = \frac{1}{2}\pi b^4$$

We can see that these volumes are equal for all *b*.

8. If f'' and g'' are continuous, and f(0) = g(0) = 0, show that

$$\int_0^a f(x)g''(x)dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx$$

$$\int_0^a f(x)g''(x)dx$$

$$\begin{array}{ll} u = f(x) & dv = g''(x)dx \\ du = f'(x)dx & v = g'(x) \end{array}$$

$$= \left[f(x)g'(x)\right]_0^a - \int_0^a f'(x)g'(x)dx$$

$$\begin{array}{ll} u = f'(x) & dv = g'(x)dx \\ du = f''(x)dx & v = g(x) \end{array}$$

$$= f(a)g'(a) - f(0)g'(0) - \left(\left[f'(x)g(x) \right]_0^a - \int_0^a f''(x)g(x)dx \right)$$

$$= f(a)g'(a) - \left(f'(a)g(a) - f'(0)g(0) - \int_0^a f''(x)g(x)dx \right)$$

$$= f(a)g'(a) - \left(f'(a)g(a) - \int_0^a f''(x)g(x)dx \right)$$

$$= f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx$$

9. A wedge is cut from an 18-inch diameter tree by first making a horizontal cut 6 inches deep, then a slanted cut starting at a point 6 inches above the horizontal cut. Set up an integral for the volume of the wedge.

We assume that the slanted plane cross the horizontal plane along a diameter of the tree. The horizontal cross-section is a disk $x^2 + y^2 = 81$. A cross-section perpendicular to the x-axis at a distance x from the origin is a right triangle, whose base is

$$b = \sqrt{81 - x^2}$$

The angle between horizontal and slanted planes is

$$\tan\theta = \frac{6}{9} = \frac{2}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\frac{2}{3}$$

Then the height of the triangle is

$$h = b\tan\theta = \sqrt{81 - x^2}\,\tan\theta = \frac{2}{3}\sqrt{81 - x^2}$$

The area of the cross-section is

$$A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{81 - x^2}\cdot\frac{2}{3}\sqrt{81 - x^2}$$

$$= \frac{1}{3}(81 - x^2)$$

The volume of the wedge is

$$V = \int_{-9}^{9} A(x)dx = \frac{1}{3}\int_{-9}^{9}(81 - x^2)dx$$