(1) Evaluate the following

(a) $\int (x^2 - 5x + 2)dx$

$$\int (x^2 - 5x + 2)dx = \int x^2 dx - 5\int x dx + 2\int dx$$
$$= \frac{1}{3}x^3 - 5 \cdot \frac{1}{2}x^2 + 2x + C$$
$$= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 2x + C$$

(b) $\int (\frac{x^2 + x}{x})\, dx$

$$\int (\frac{x^2 + x}{x})\, dx = \int \left(\frac{x^2}{x} + \frac{x}{x} \right) dx$$
$$= \int (x + 1)\, dx$$
$$= \int x\, dx + \int dx$$
$$= \frac{1}{2}x^2 + x + C$$

(c) $\int (6x^2 \cos(x^3))\, dx$

$$\int (6x^2 \cos(x^3))\, dx = 6\int x^2 \cos(x^3)\, dx$$

$u = x^3$

$du = 3x^2 dx$

$$= 6\int \frac{1}{3} \cos u\, du$$
$$= 2\int \cos u\, du$$
$$= 2\sin u + C$$
$$= 2\sin x^3 + C$$

(d) $\int (x\sqrt{x-1})\,dx$

$$\int (x\sqrt{x-1})\,dx$$

$$\left.\begin{aligned} u &= x-1 \\ du &= dx \end{aligned}\right\|$$

$$= \int (u+1)\sqrt{u}\,du = \int (u\sqrt{u} + \sqrt{u})\,du$$

$$= \int (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du = \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-1)^2\sqrt{x-1} + \frac{2}{3}(x-1)\sqrt{x-1} + C$$

(e) $\int_0^{\pi} \sin(2x)\,dx$

$$\int_0^{\pi} \sin(2x)\,dx$$

$$\left.\begin{array}{llll} u = 2x & x & 0 & \pi \\ du = 2dx & u & 0 & 2\pi \end{array}\right\|$$

$$= \int_0^{2\pi} \frac{1}{2}\sin u\,du = \left[-\frac{1}{2}\cos u\right]_0^{2\pi}$$

$$= -\frac{1}{2}\cos 2\pi + -\frac{1}{2}\cos 0$$

$$= -\frac{1}{2}\cdot 1 + -\frac{1}{2}\cdot 1$$

$$= 0$$

(2) Find the area under the "hump" of the cos(x) on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The area is

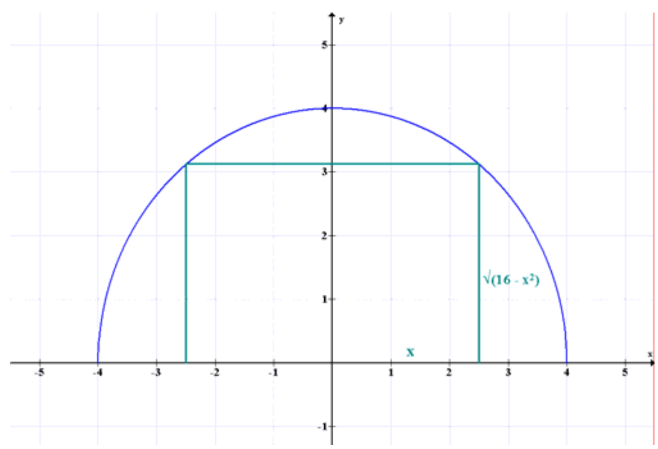
$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 1 + 1$$

$$= 2$$

(3) A rectangle is to be inscribed in a semicircle of radius 4. What is the largest area the rectangle can have and what are its dimensions?

Let the circle be centered at the origin. Then its equation is
$x^2 + y^2 = 16$
The equation of the upper semicircle is
$y = \sqrt{16 - x^2}$
Let 2x be the length of the rectangle.
Using the dimensions labeled in the figure, the base of the rectangle is 2x and its height is $\sqrt{16 - x^2}$.
The area of the rectangle is

$A(x) = 2x\sqrt{16 - x^2} \qquad 0 \le x \le 4$



Find the critical numbers

$$A'(x) = 2\sqrt{16 - x^2} + 2x \cdot \frac{1}{2}(16 - x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= 2\sqrt{16 - x^2} - \frac{2x^2}{\sqrt{16 - x^2}}$$

$$= \frac{2(16 - x^2) - 2x^2}{\sqrt{16 - x^2}}$$

$$= \frac{32 - 4x^2}{\sqrt{16 - x^2}}$$

The derivative is undefined at x = 0 and x = ±4.

$A'(x) = 0$

$$\frac{32 - 4x^2}{\sqrt{16 - x^2}} = 0$$

$32 - 4x^2 = 0$

$x^2 = 8$

$x = 2\sqrt{2}$

The critical numbers are 0, $2\sqrt{2}$ and 4 (2x is the length, so it must be positive).

$A(0) = 2\cdot 0\sqrt{16-0^2} = 0$

$A(4) = 2\cdot 4\sqrt{16-4^2} = 0$

$A(2\sqrt{2}) = 2\cdot 2\sqrt{2}\sqrt{16-(2\sqrt{2})^2} = 4\sqrt{2}\sqrt{8} = 16$

Thus, the largest area is 16.

The length is $2x = 2\cdot 2\sqrt{2} = 4\sqrt{2}$ and the height is $\sqrt{16-(2\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$

(4) Find the area between the graphs of the functions $f(x) = 3x^2 - x$ and $g(x) = x^3 + x$.

Find the x-coordinates of the points of intersection:

$3x^2 - x = x^3 + x$

$0 = x^3 + x + x - 3x^2$

$0 = x^3 - 3x^2 + 2x$

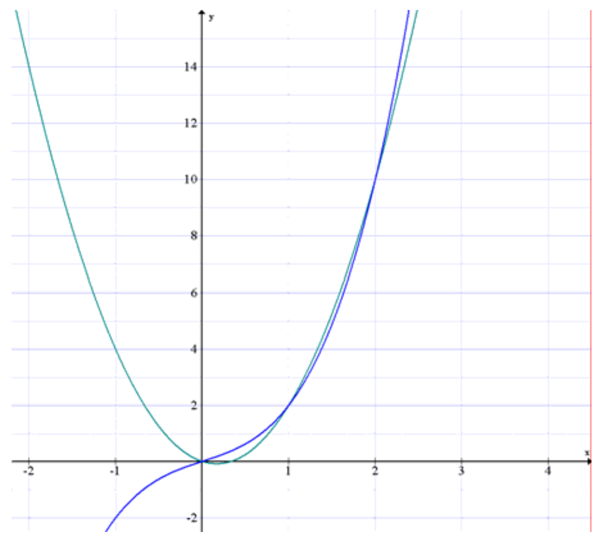
$0 = x(x^2 - 3x + 2)$

$0 = x(x^2 - x - 2x + 2)$

$0 = x(x(x-1) - 2(x-1))$

$0 = x(x-1))(x-2)$

x = 0 or x = 1 or x = 2



The area is

$$A = \int_0^1 (g(x) - f(x))\,dx + \int_1^2 (f(x) - g(x))\,dx$$

$$= \int_0^1 (x^3 + x - (3x^2 - x))\,dx + \int_1^2 (3x^2 - x - (x^3 + x))\,dx$$

$$= \int_0^1 (x^3 + x - 3x^2 + x)\,dx + \int_1^2 (3x^2 - x - x^3 - x)\,dx$$

$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx + \int_1^2 (3x^2 - 2x - x^3)\,dx$$

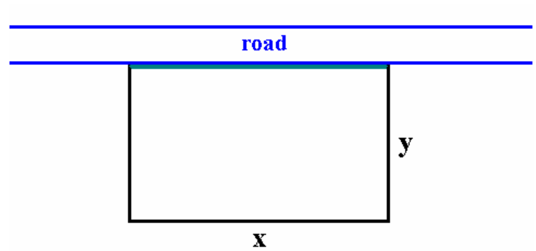
$$= \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^1 + \left[x^3 - x^2 - \frac{1}{4}x^4\right]_1^2$$

$$= \frac{1}{4}\cdot 1^4 - 1^3 + 1^2 + 2^3 - 2^2 - \frac{1}{4}\cdot 2^4 - \left(1^3 - 1^2 - \frac{1}{4}\cdot 1^4\right)$$

$$= \frac{1}{4} - 1 + 1 + 8 - 4 - 4 - \left(1 - 1 - \frac{1}{4}\right)$$

$$= \frac{1}{2}$$

(5) A rancher plans to set aside a rectangular region of one square kilometer for cattle and wishes to build a wooden fence to enclose the region. Since one side of the region will run along the road, the rancher decides to use a better quality wood for that side which costs three times as much as the wood for the other sides. What dimensions will minimize the cost of the fence?



Let x be the length of the rectangle and let y be the width of the rectangle. Let *a* be the cost of 1 *km* of usual quality wood. Then the cost of 1 *km* a better quality wood is 3*a*.

The total cost of fence is

C = (x + 2y)*a* + 3*a*x = 2*ay* + 4*ax*

The area of the rectangle is

A = xy = 1

Then $y = \frac{1}{x}$ and

$$C(x) = \frac{2a}{x} + 4ax$$

Find the critical numbers:

$$C'(x) = -\frac{2a}{x^2} + 4a$$

The critical numbers can occur only when $C'(x) = 0$ (x = 0 is not in the domain of C(x)).

$$-\frac{2a}{x^2} + 4a = 0$$

$$-2a + 4ax^2 = 0$$

$$-2a(1 - 2x^2) = 0$$

$$1 - 2x^2 = 0$$

$$x^2 = \frac{1}{2}$$

$x = \frac{\sqrt{2}}{2}$ (the length cannot be negative)

$$y = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)} = \sqrt{2}$$

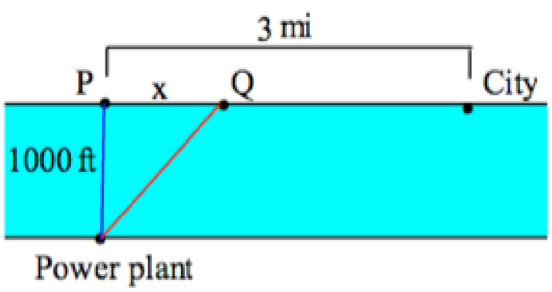
$$\lim_{x\to 0} C(x) = \lim_{x\to 0}\left(\frac{2a}{x} + 4ax\right) = \lim_{x\to 0}\frac{2a}{x} + \lim_{x\to 0} 4ax = \lim_{x\to 0}\frac{2a}{x} + 0 = \lim_{x\to 0}\frac{2a}{x} = \infty$$

$$C\left(\frac{\sqrt{2}}{2}\right) = \frac{2a}{\frac{\sqrt{2}}{2}} + 4a\cdot\frac{\sqrt{2}}{2} = 4\sqrt{2}a$$

$$\lim_{x\to\infty} C(x) = \lim_{x\to\infty}\left(\frac{2a}{x} + 4ax\right) = \lim_{x\to\infty}\frac{2a}{x} + \lim_{x\to\infty}(4ax) = 0 + \lim_{x\to\infty}(4ax) = \infty$$

Answer: The cost will have a minimum when the length of the side along the road must be $\frac{\sqrt{2}}{2}$ km and the length of the perpendicular side must be $\sqrt{2}$ km.

(6) A power company has a power plant on a river that is 1000 feet wide. To lay a new cable from the plant to a city three miles downstream (see drawing) on the opposite side of the river cost \$160 per foot across the river and \$80 per foot across the land. Find the value of x (the distance from point P to Q) that minimizes the cost of laying the cable.



The distance from the Power plant to the point Q is $\sqrt{1000^2 + x^2} = \sqrt{1000000 + x^2}$ so the cost of laying the cable is

$160\sqrt{1000^2 + x^2}$

The distance from the point Q to the City is

3(5280) – x = 15840 – x

the cost of laying the cable is

80(15840 – x) = 1267200 – 80x.

The total cost is

$C(x) = 160\sqrt{1000000 + x^2} + 1267200 - 80x \qquad 0 \le x \le 15840$

Find the critical numbers

$$C'(x) = 160 \cdot \frac{1}{2}(1000000 + x^2)^{-\frac{1}{2}} \cdot 2x - 80 = \frac{160x}{\sqrt{1000000 + x^2}} - 80$$

$C'(x)$ is defined for all x so the critical numbers can occur only when $C'(x) = 0$:

$$\frac{160x}{\sqrt{1000000+x^2}}-80=0$$
$$\frac{160x}{\sqrt{1000000+x^2}}=80$$
$$2x=\sqrt{1000000+x^2}$$
$$4x^2=1000000+x^2$$
$$3x^2=1000000$$
$$x^2=\frac{1000000}{3}$$
$$x=\sqrt{\frac{1000000}{3}}$$
$$x=\frac{1000}{\sqrt{3}}$$
$$x=\frac{1000\sqrt{3}}{3}$$
$$C(0)=160\sqrt{1000000+0^2}+1267200-80\cdot 0=1427200$$
$$C\left(\frac{1000\sqrt{3}}{3}\right)=160\sqrt{1000000+\left(\frac{1000\sqrt{3}}{3}\right)^2}+1267200-80\left(\frac{1000\sqrt{3}}{3}\right)\approx 1405764$$
$$C(15840)=160\sqrt{1000000+15840^2}+1267200-80\cdot 15840\approx 2539445$$

Answer: The cost will have the minimum when $x=\frac{1000\sqrt{3}}{3}$.