

(1) Evaluate the following

(a)  $\int (x^2 - 5x + 2) dx$

$$\begin{aligned}\int (x^2 - 5x + 2) dx &= \int x^2 dx - 5 \int x dx + 2 \int dx \\ &= \frac{1}{3} x^3 - 5 \cdot \frac{1}{2} x^2 + 2x + C \\ &= \frac{1}{3} x^3 - \frac{5}{2} x^2 + 2x + C\end{aligned}$$

(b)  $\int \left( \frac{x^2 + x}{x} \right) dx$

$$\begin{aligned}\int \left( \frac{x^2 + x}{x} \right) dx &= \int \left( \frac{x^2}{x} + \frac{x}{x} \right) dx \\ &= \int (x + 1) dx \\ &= \int x dx + \int dx \\ &= \frac{1}{2} x^2 + x + C\end{aligned}$$

(c)  $\int (6x^2 \cos(x^3)) dx$

$$\begin{aligned}\int (6x^2 \cos(x^3)) dx &= 6 \int x^2 \cos(x^3) dx \\ u = x^3 &\parallel \\ du = 3x^2 dx &\parallel \\ &= 6 \int \frac{1}{3} \cos u du \\ &= 2 \int \cos u du \\ &= 2 \sin u + C \\ &= 2 \sin x^3 + C\end{aligned}$$

$$(d) \int (x\sqrt{x-1}) dx$$

$$\int (x\sqrt{x-1}) dx$$

$$u = x-1 \parallel$$

$$du = dx \parallel$$

$$= \int (u+1)\sqrt{u} du = \int (u\sqrt{u} + \sqrt{u}) du$$

$$= \int (u^{\frac{3}{2}} + u^{\frac{1}{2}}) du = \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-1)^2\sqrt{x-1} + \frac{2}{3}(x-1)\sqrt{x-1} + C$$

$$(e) \int_0^{\pi} \sin(2x) dx$$

$$\int_0^{\pi} \sin(2x) dx$$

$$u = 2x \quad x \quad 0 \quad \pi \parallel$$

$$du = 2dx \quad u \quad 0 \quad 2\pi \parallel$$

$$= \int_0^{2\pi} \frac{1}{2} \sin u du = \left[ -\frac{1}{2} \cos u \right]_0^{2\pi}$$

$$= -\frac{1}{2} \cos 2\pi + \frac{1}{2} \cos 0$$

$$= -\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1$$

$$= 0$$

- (2) Find the area under the “hump” of the  $\cos(x)$  on the interval  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ .

The area is

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = \left[ \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin \frac{\pi}{2} - \sin \left( -\frac{\pi}{2} \right) = \sin \frac{\pi}{2} + \sin \frac{\pi}{2} = 1 + 1 = 2$$

- (3) A rectangle is to be inscribed in a semicircle of radius 4. What is the largest area the rectangle can have and what are its dimensions?

Let the circle be centered at the origin. Then its equation is

$$x^2 + y^2 = 16$$

The equation of the upper semicircle is

$$y = \sqrt{16 - x^2}$$

Let  $2x$  be the length of the rectangle.

Using the dimensions labeled in the figure, the base of the rectangle is  $2x$  and its height is  $\sqrt{16 - x^2}$ .

The area of the rectangle is

$$A(x) = 2x\sqrt{16 - x^2} \quad 0 \leq x \leq 4$$

Find the critical numbers

$$A'(x) = 2\sqrt{16 - x^2} + 2x \cdot \frac{1}{2}(16 - x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$= 2\sqrt{16 - x^2} - \frac{2x^2}{\sqrt{16 - x^2}}$$

$$= \frac{2(16 - x^2) - 2x^2}{\sqrt{16 - x^2}}$$

$$= \frac{32 - 4x^2}{\sqrt{16 - x^2}}$$

The derivative is undefined at  $x = 0$  and  $x = \pm 4$ .

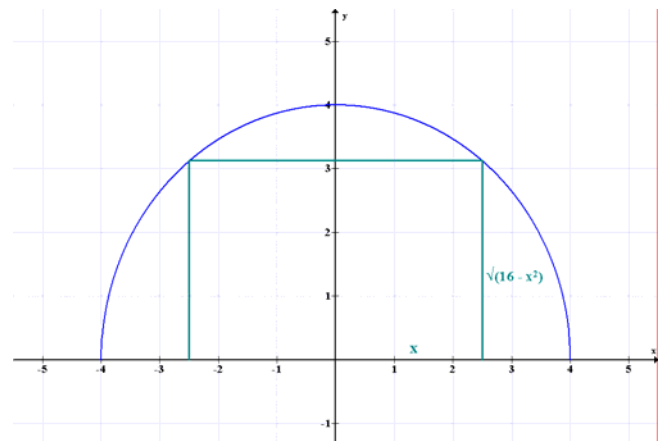
$$A'(x) = 0$$

$$\frac{32 - 4x^2}{\sqrt{16 - x^2}} = 0$$

$$32 - 4x^2 = 0$$

$$x^2 = 8$$

$$x = 2\sqrt{2}$$



The critical numbers are 0,  $2\sqrt{2}$  and 4 ( $2x$  is the length, so it must be positive).

$$A(0) = 2 \cdot 0 \sqrt{16 - 0^2} = 0$$

$$A(4) = 2 \cdot 4 \sqrt{16 - 4^2} = 0$$

$$A(2\sqrt{2}) = 2 \cdot 2\sqrt{2} \sqrt{16 - (2\sqrt{2})^2} = 4\sqrt{2} \sqrt{8} = 16$$

Thus, the largest area is 16.

The length is  $2x = 2 \cdot 2\sqrt{2} = 4\sqrt{2}$  and the height is  $\sqrt{16 - (2\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$

(4) Find the area between the graphs of the functions  $f(x) = 3x^2 - x$  and  $g(x) = x^3 + x$ .

Find the x-coordinates of the points of intersection:

$$3x^2 - x = x^3 + x$$

$$0 = x^3 + x + x - 3x^2$$

$$0 = x^3 - 3x^2 + 2x$$

$$0 = x(x^2 - 3x + 2)$$

$$0 = x(x^2 - x - 2x + 2)$$

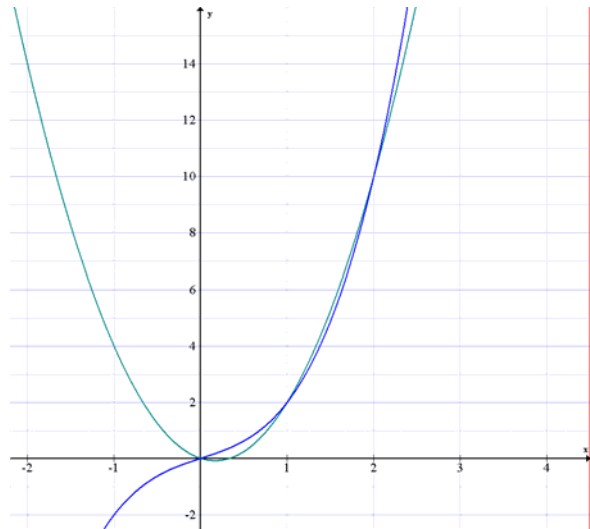
$$0 = x(x(x - 1) - 2(x - 1))$$

$$0 = x(x - 1)(x - 2)$$

$$x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = 2$$

The area is

$$\begin{aligned} A &= \int_0^1 (g(x) - f(x)) dx + \int_1^2 (f(x) - g(x)) dx \\ &= \int_0^1 (x^3 + x - (3x^2 - x)) dx + \int_1^2 (3x^2 - x - (x^3 + x)) dx \\ &= \int_0^1 (x^3 + x - 3x^2 + x) dx + \int_1^2 (3x^2 - x - x^3 - x) dx \\ &= \int_0^1 (x^3 - 3x^2 + 2x) dx + \int_1^2 (3x^2 - 2x - x^3) dx \\ &= \left[ \frac{1}{4}x^4 - x^3 + x^2 \right]_0^1 + \left[ x^3 - x^2 - \frac{1}{4}x^4 \right]_1^2 \\ &= \frac{1}{4} \cdot 1^4 - 1^3 + 1^2 + 2^3 - 2^2 - \frac{1}{4} \cdot 2^4 - \left( 1^3 - 1^2 - \frac{1}{4} \cdot 1^4 \right) \\ &= \frac{1}{4} - 1 + 1 + 8 - 4 - 4 - \left( 1 - 1 - \frac{1}{4} \right) \\ &= \frac{1}{2} \end{aligned}$$



- (5) A rancher plans to set aside a rectangular region of one square kilometer for cattle and wishes to build a wooden fence to enclose the region. Since one side of the region will run along the road, the rancher decides to use a better quality wood for that side which costs three times as much as the wood for the other sides. What dimensions will minimize the cost of the fence?

Let  $x$  be the length of the rectangle and let  $y$  be the width of the rectangle. Let  $a$  be the cost of 1 km of usual quality wood. Then the cost of 1 km a better quality wood is  $3a$ .

The total cost of fence is

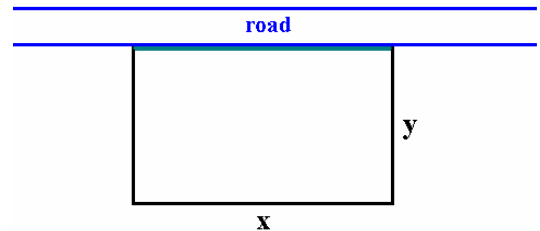
$$C = (x + 2y)a + 3ax = 2ay + 4ax$$

The area of the rectangle is

$$A = xy = 1$$

Then  $y = \frac{1}{x}$  and

$$C(x) = \frac{2a}{x} + 4ax$$



Find the critical numbers:

$$C'(x) = -\frac{2a}{x^2} + 4a$$

The critical numbers can occur only when  $C'(x) = 0$  ( $x = 0$  is not in the domain of  $C(x)$ ).

$$-\frac{2a}{x^2} + 4a = 0$$

$$-2a + 4ax^2 = 0$$

$$-2a(1 - 2x^2) = 0$$

$$1 - 2x^2 = 0$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2} \quad (\text{the length cannot be negative})$$

$$y = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)} = \sqrt{2}$$

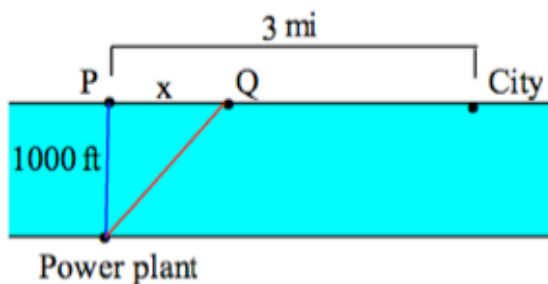
$$\lim_{x \rightarrow 0} C(x) = \lim_{x \rightarrow 0} \left( \frac{2a}{x} + 4ax \right) = \lim_{x \rightarrow 0} \frac{2a}{x} + \lim_{x \rightarrow 0} 4ax = \lim_{x \rightarrow 0} \frac{2a}{x} + 0 = \lim_{x \rightarrow 0} \frac{2a}{x} = \infty$$

$$C\left(\frac{\sqrt{2}}{2}\right) = \frac{2a}{\frac{\sqrt{2}}{2}} + 4a \cdot \frac{\sqrt{2}}{2} = 4\sqrt{2}a$$

$$\lim_{x \rightarrow \infty} C(x) = \lim_{x \rightarrow \infty} \left( \frac{2a}{x} + 4ax \right) = \lim_{x \rightarrow \infty} \frac{2a}{x} + \lim_{x \rightarrow \infty} (4ax) = 0 + \lim_{x \rightarrow \infty} (4ax) = \infty$$

Answer: The cost will have a minimum when the length of the side along the road must be  $\frac{\sqrt{2}}{2}$  km and the length of the perpendicular side must be  $\sqrt{2}$  km.

- (6) A power company has a power plant on a river that is 1000 feet wide. To lay a new cable from the plant to a city three miles downstream (see drawing) on the opposite side of the river cost \$160 per foot across the river and \$80 per foot across the land. Find the value of  $x$  (the distance from point P to Q) that minimizes the cost of laying the cable.



The distance from the Power plant to the point Q is  $\sqrt{1000^2 + x^2} = \sqrt{1000000 + x^2}$  so the cost of laying the cable is

$$160\sqrt{1000^2 + x^2}$$

The distance from the point Q to the City is

$$3(5280) - x = 15840 - x$$

the cost of laying the cable is

$$80(15840 - x) = 1267200 - 80x.$$

The total cost is

$$C(x) = 160\sqrt{1000000 + x^2} + 1267200 - 80x \quad 0 \leq x \leq 15840$$

Find the critical numbers

$$C'(x) = 160 \cdot \frac{1}{2} (1000000 + x^2)^{-\frac{1}{2}} \cdot 2x - 80 = \frac{160x}{\sqrt{1000000 + x^2}} - 80$$

$C'(x)$  is defined for all  $x$  so the critical numbers can occur only when  $C'(x) = 0$ :

$$\frac{160x}{\sqrt{1000000 + x^2}} - 80 = 0$$

$$\frac{160x}{\sqrt{1000000 + x^2}} = 80$$

$$2x = \sqrt{1000000 + x^2}$$

$$4x^2 = 1000000 + x^2$$

$$3x^2 = 1000000$$

$$x^2 = \frac{1000000}{3}$$

$$x = \sqrt{\frac{1000000}{3}}$$

$$x = \frac{1000}{\sqrt{3}}$$

$$x = \frac{1000\sqrt{3}}{3}$$

$$C(0) = 160\sqrt{1000000 + 0^2} + 1267200 - 80 \cdot 0 = 1427200$$

$$C\left(\frac{1000\sqrt{3}}{3}\right) = 160\sqrt{1000000 + \left(\frac{1000\sqrt{3}}{3}\right)^2} + 1267200 - 80\left(\frac{1000\sqrt{3}}{3}\right) \approx 1405764$$

$$C(15840) = 160\sqrt{1000000 + 15840^2} + 1267200 - 80 \cdot 15840 \approx 2539445$$

Answer: The cost will have the minimum when  $x = \frac{1000\sqrt{3}}{3}$ .