1. $\frac{dx}{dy} = -\frac{4y^2 + 6xy}{3y^2 + 2x}$, y(0) = 3

$$\frac{dx}{dy} = -\frac{4y^2 + 6xy}{3y^2 + 2x}$$

$$(3y^2 + 2x)dx = -(4y^2 + 6xy)dy$$

$$(3y^2 + 2x)dx + (4y^2 + 6xy)dy = 0$$

This equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

with $M(x, y) = 3y^2 + 2x$ and $N(x, y) = 4y^2 + 6xy$.

We have

$$\frac{\partial M}{\partial y} = 6y \text{ and } \frac{\partial N}{\partial x} = 6y$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then we have an exact equation. Thus there is $\psi(x, y)$ such that

$$\psi_x(x, y) = M(x, y)$$

$$\psi_y(x, y) = N(x, y)$$

Then

$$\psi(x, y) = \int M(x, y)dx = \int (3y^2 + 2x)dx = 3y^2 x + x^2 + C(y)$$

Setting $\psi_y(x, y) = N(x, y)$ we get

$$6yx + C'(y) = 4y^2 + 6xy$$

$$C'(y) = 4y^2$$

$$C(y) = \int 4y^2 dy = \frac{4}{3}y^3$$

Thus

$$\psi(x, y) = 3y^2 x + x^2 + \frac{4}{3}y^3$$

The solution to the differential equation is

$$3y^2 x + x^2 + \frac{4}{3}y^3 = C$$

Find the constant C:

$$3 \cdot 3^2 \cdot 0 + 0^2 + \frac{4}{3} \cdot 3^3 = C$$

$$36 = C$$

Therefore

$$3y^2 x + x^2 + \frac{4}{3}y^3 = 36$$

2. $(x^2 + y^2 - 5)dx = (y + xy)dy$

a) Rewrite the equation

$$(x^2 + y^2 - 5)dx - (y + xy)dy = 0 \qquad (1)$$

This equation is of the form

$M(x,y)dx + N(x,y)dy = 0$

with $M(x,y) = x^2 + y^2 - 5$ and $N(x,y) = -y - xy$.

We have

$$\frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = -y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then the equation is not exact.

b) Since

$$\frac{\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}}{N(x,y)} = \frac{2y-(-y)}{-y-xy} = \frac{3y}{-y(1+x)} = -\frac{3}{1+x} \equiv p(x)$$

then there is an integrating factor μ that depends on *x* only.

$$\mu = e^{\int p(x)dx} = e^{-\int \frac{3}{1+x}dx} = e^{-3\ln|x+1|} = e^{(\ln|x+1|)^{-3}} = (x+1)^{-3} = \frac{1}{(x+1)^3}$$

Multiply each side of (1) by μ:

$$\frac{1}{(x+1)^3}(x^2+y^2-5)dx - \frac{1}{(x+1)^3}(y+xy)dy = 0$$

$$\frac{x^2+y^2-5}{(x+1)^3}dx - \frac{y}{(x+1)^2}dy = 0$$

We have

$$M(x,y) = \frac{x^2+y^2-5}{(x+1)^3} \quad \Rightarrow \quad M_y(x,y) = \frac{2y}{(x+1)^3}$$

$$N(x,y) = -\frac{y}{(x+1)^2} \quad \Rightarrow \quad N_x(x,y) = \frac{2y}{(x+1)^3}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then we have an exact equation. Thus there is $\psi(x,y)$ such that

$$\psi_x(x,y) = M(x,y)$$

$$\psi_y(x,y) = N(x,y)$$

Then

$$\psi(x,y) = \int M(x,y)dx = \int \frac{x^2+y^2-5}{(x+1)^3}dx$$

$$= \int \frac{x^2}{(x+1)^3}dx + \int \frac{y^2-5}{(x+1)^3}dx$$

$$\left.\begin{array}{l} z = x+1 \\ dz = dx \end{array}\right\|$$

$$= \int \frac{(z-1)^2}{z^3}dz + \int \frac{y^2-5}{z^3}dz$$

$$= \int \frac{z^2-2z+1}{z^3}dz - \frac{y^2-5}{2z^2}$$

$$= \int \left(\frac{1}{z} - \frac{2}{z^2} + \frac{1}{z^3}\right)dz - \frac{y^2-5}{2z^2}$$

$$= \ln|z| + \frac{2}{z} - \frac{1}{2z^2} - \frac{y^2-5}{2z^2} + C(y)$$

$$= \ln|z| + \frac{2}{z} - \frac{y^2-4}{2z^2} + C(y)$$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{y^2-4}{2(x+1)^2} + C(y)$$

Setting $\psi_y(x,y) = N(x,y)$ we get

$$\psi_y(x,y) = -\frac{y}{(x+1)^2} + C'(y) = -\frac{y}{(x+1)^2}$$

$$C'(y) = 0$$

$$C(y) = K$$

Thus

$$\psi(x,y) = \ln|x+1| + \frac{2}{x+1} - \frac{y^2-4}{2(x+1)^2} + K$$

The solution to the differential equation is

$$\ln|x+1| + \frac{2}{x+1} - \frac{y^2-4}{2(x+1)^2} = C$$

3. a) $(2x+y+1)y' = 1$

Use the substitution

v = 2x + y $\Rightarrow$ v′ = 2 + y′

Then

$$(v+1)(v'-2) = 1$$

$$v' - 2 = \frac{1}{v+1}$$

$$v' = \frac{1}{v+1} + 2$$

$$\frac{dv}{dx} = \frac{1}{v+1} + 2$$

$$\frac{dv}{\frac{1}{v+1} + 2} = dx$$

$$\frac{(v+1)}{1+2(v+1)}dv = dx$$

$$\frac{v+1}{2v+3}dv = dx$$

$$\frac{1}{2}\frac{v+1}{v+\frac{3}{2}}dv = dx$$

$$\frac{1}{2}\frac{v+\frac{3}{2}-\frac{1}{2}}{v+\frac{3}{2}}dv = dx$$

$$\frac{1}{2}\left(1-\frac{\frac{1}{2}}{v+\frac{3}{2}}\right)dv = dx$$

$$\frac{1}{2}(v-\frac{1}{2}\ln|v+\frac{3}{2}|) = x + C_1$$

$$2x + y - \frac{1}{2}\ln\left(2x + y + \frac{3}{2}\right) = 2x + 2C_1$$

$$y - \frac{1}{2}\ln\left(2x + y + \frac{3}{2}\right) = C_2$$ $C_2 = 2C_1$

$$2y - \ln\left(2x + y + \frac{3}{2}\right) = 2C_2$$

$$2y - \ln\left(2x + y + \frac{3}{2}\right) = C$$ $C = 2C_2$

b) $3(1+t^2)\frac{dy}{dt} = 2ty(y^3-1)$

$$\frac{1}{y(y^3-1)}dy = \frac{2t}{3(1+t^2)}dt$$

Use the substitution

$v = 1 + t^2$ $\Rightarrow$ $\frac{dv}{dt} = 2t$ $\Rightarrow$ $dv = 2tdt$

Then

$$\frac{1}{y(y^3-1)}dy = \frac{1}{3v}dv$$

$$\frac{1}{y(y-1)(y^2+y+1)}dy = \frac{1}{3v}dv \qquad (1)$$

Use the partial fraction decomposition

$$\frac{1}{y(y-1)(y^2+y+1)} = \frac{A}{y} + \frac{B}{y-1} + \frac{Cy+D}{y^2+y+1}$$

$$1 = A(y^3-1) + By(y^2+y+1) + (Cy+D)y(y-1)$$
$$1 = Ay^3 - A + By^3 + By^2 + By + Cy^3 - Cy^2 + Dy^2 - Dy$$
$$1 = (A+B+C)y^3 + (B-C+D)y^2 + (B-D)y - A$$

$$\begin{cases} A+B+C=0 \\ B-C+D=0 \\ B-D=0 \\ -A=1 \end{cases} \Rightarrow \begin{cases} -1+B+C=0 \\ B-C+D=0 \\ B-D=0 \\ A=-1 \end{cases} \Rightarrow \begin{cases} B=\frac{1}{3} \\ B-C+D=0 \\ B-D=0 \\ A=-1 \end{cases}$$

$$\Rightarrow \begin{cases} B=\frac{1}{3} \\ \frac{1}{3}-C+D=0 \\ \frac{1}{3}-D=0 \\ A=-1 \end{cases} \Rightarrow \begin{cases} B=\frac{1}{3} \\ \frac{1}{3}-C+\frac{1}{3}=0 \\ D=\frac{1}{3} \\ A=-1 \end{cases} \Rightarrow \begin{cases} B=\frac{1}{3} \\ C=\frac{2}{3} \\ D=\frac{1}{3} \\ A=-1 \end{cases}$$

Then

$$\frac{1}{y(y-1)(y^2+y+1)}=-\frac{1}{y}+\frac{\frac{1}{3}}{y-1}+\frac{\frac{2}{3}y+\frac{1}{3}}{y^2+y+1}$$

Substitute this in (1)

$$\left(-\frac{1}{y}+\frac{\frac{1}{3}}{y-1}+\frac{\frac{2}{3}y+\frac{1}{3}}{y^2+y+1}\right)dy=\frac{1}{3v}dv$$

$$\left(-\frac{1}{y}+\frac{\frac{1}{3}}{y-1}+\frac{1}{3}\frac{2y+1}{y^2+y+1}\right)dy=\frac{1}{3v}dv$$

$$\int\left(-\frac{1}{y}+\frac{\frac{1}{3}}{y-1}+\frac{1}{3}\frac{2y+1}{y^2+y+1}\right)dy=\int\frac{1}{3v}dv$$

$$-\int\frac{1}{y}dy+\frac{1}{3}\int\frac{1}{y-1}dy+\frac{1}{3}\int\frac{2y+1}{y^2+y+1}dy=\int\frac{1}{3v}dv$$

$$-\ln|y|+\frac{1}{3}\ln|y-1|+\frac{1}{3}\int\frac{d(y^2+y)}{y^2+y+1}=\frac{1}{3}\ln|v|$$

$$-\ln|y|+\frac{1}{3}\ln|y-1|+\frac{1}{3}\ln|y^2+y+1|=\frac{1}{3}\ln|v|+\frac{1}{3}\ln|C|$$

$$-3\ln|y|+\ln|y-1|+\ln|y^2+y+1|=\ln|v|+\ln|C|$$

$$\ln|y^{-3}|+\ln|(y-1)(y^2+y+1)|=\ln|Cv|$$

$$\frac{(y-1)(y^2+y+1)}{y^3}=Cv$$

$$\frac{y^3-1}{y^3}=Cv$$

$$1-\frac{1}{y^3}=Cv$$

$$1-Cv=\frac{1}{y^3}$$

$$y^3=\frac{1}{1-Cv}$$

$$y = \frac{1}{\sqrt[3]{1 - Cv}}$$

$$y = \frac{1}{\sqrt[3]{1 - C(1 + t^2)}}$$

c) $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2},\ x > 0$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{x^2 - y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \frac{y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$$

Use the substitution

$$y = ux \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{du}{dx}x + u$$

$$\frac{du}{dx}x + u = u + \sqrt{1 - u^2}$$

$$\frac{du}{dx}x = \sqrt{1 - u^2}$$

$$\frac{du}{\sqrt{1 - u^2}} = \frac{dx}{x}$$

$$\int \frac{du}{\sqrt{1 - u^2}} = \int \frac{dx}{x}$$

$\sin^{-1} u = \ln x + C$ (since x > 0)

$$\sin(\sin^{-1} u) = \sin(\ln x + C)$$

$$u = \sin(\ln x + C)$$

$$\frac{y}{x} = \sin(\ln x + C)$$

$$y = x\sin(\ln x + C)$$